

# Photon spin operator and Pauli matrix

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(Dated: November 17, 2010)

## Abstract

Any polarization vector of a plane wave can be decomposed into a pair of mutually orthogonal base vectors, known as a polarization basis. Regarding this decomposition as a quasi-unitary transformation from a three-component vector to a corresponding two-component spinor, one is led to a representation formalism for the photon spin. The spin operator  $\hat{\gamma}$  defined on the space of unit spinors, referred to as the Jones space, has only component along the wave vector and is represented by one of the Pauli matrices in the commonly used polarization basis. It is deformed by the quasi-unitary transformation from the spin operator that is defined on the space of unit polarization vectors, referred to as the Pancharatnam space. On the basis of this theory, it is shown that the Cartesian components of spin operator  $\hat{\gamma}$  are mutually commutative and the spin angular momentum in units of  $\hbar$  is exactly the component of the Stokes vector along the wave vector.

PACS numbers: 42.50.Tx, 02.10.Yn, 42.25.Ja

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## I. INTRODUCTION

The spin angular momentum (SAM) of the photon is a basic quantity in the nature. Since it was conceived [1] and experimentally detected [2, 3], the photon spin has been puzzling and still remains mysterious for the time being. First of all, the separability of the spin from the orbital angular momentum has been a controversial issue for a long time [4–12]. The key point for this is usually attributed [4, 5, 9] to the transversality of the radiation field. But there appeared in the past decade more and more experimental evidences that the SAM is different from the orbital angular momentum. The spin and orbital angular momentum have distinct effects on tiny birefringent particles held in optical tweezers [13]. The conversion of the SAM to the orbital angular momentum was observed in anisotropic [14], isotropic [15] and nonlinear [16] media. The spin-orbit interaction of a photon was also detected [17–19].

Secondly, a common opinion as for the relation between the spin and the polarization is that the spin is the polarization itself [5, 8, 20–22] or is, ambiguously, associated with the polarization [12, 23, 24]. This seems all that we can say about the relation between the spin and the polarization. But we are faced with a dilemma. On one hand, it was observed [10, 11] that the Cartesian components of the spin operator in a second quantized theory commute with one another. On the other hand, the photon polarization is usually taken as a classical analogue in quantum mechanics [25] to illustrate the non-commutability of the Cartesian components of the spin of the electron. The question arises naturally as to how to understand the aforementioned commutativity between the Cartesian components of the spin operator.

More importantly, though the photon spin is equal to 1, it has only two independent eigen components corresponding to the left-handed and right-handed circular polarizations. So the Pauli matrices are frequently used for the spin operator in either theoretical or experimental works, concerning, for example, the SAM transfer [26, 27] and the spin Hall effect [17, 28, 29]. But it is not without problem [30] to regard the Pauli matrices as the Cartesian components of the spin operator. In the first place, they do not obey the commutation relations appropriate for a spin-1 particle. In the second place, they are not in consistency with van Enk and Nienhuis' observation [10, 11] that the Cartesian components of the spin operator are commutative with one another.

The purpose of this paper is to investigate whether and how the spin operator of the photon can be expressed in terms of the Pauli matrix. As we know [4, 10, 11, 30], the spin operator that obeys the commutation relations appropriate for a spin-1 particle is given by 3-by-3 matrices  $\hat{\mathbf{S}} = \hbar \hat{\boldsymbol{\Sigma}}$ , which act on three-component vectors referred to as the photon wave function, where  $(\hat{\boldsymbol{\Sigma}}_k)_{ij} = -i\epsilon_{ijk}$  with  $\epsilon_{ijk}$  the Levi-Civita pseudotensor. It will be shown that the previously introduced 3-by-2 matrix [31] on the basis of transversality condition is a quasi-unitary transformation (QUT). The spin operator  $\hat{\mathbf{S}}$  defined on the space of three-component vectors is transformed by this QUT into one that is defined on the space of two-component spinors. It is the transformed spin operator that is expressible in terms of the Pauli matrix, but in quite an unusual way. We will see that (i) the spin operator on the spinor space has only component along the wave vector, so that its Cartesian components commute with one another; (ii) the spin operator on the spinor space generates a rotation of the spinor about the wave vector; (iii) the SAM in units of  $\hbar$  is exactly the component of the Stokes vector along the wave vector.

## II. SPIN OPERATOR EXPRESSED IN TERMS OF ONE PAULI MATRIX

Consider in free space an arbitrary normalized monochromatic radiation field. The photon wave function [4]  $\mathbf{E}(\mathbf{k})$  in momentum representation satisfies  $\int \mathbf{E}^\dagger \mathbf{E} d\Omega = 1$ , where the boldfaced symbol stands for a column vector of three components, the superscript  $\dagger$  denotes the conjugate transpose, and  $\mathbf{b}^\dagger \mathbf{a}$  means the inner product  $\mathbf{b}^* \cdot \mathbf{a}$  of two complex vectors  $\mathbf{a}$  and  $\mathbf{b}$ . Its SAM was rigorously separated very recently [32] from its orbital angular momentum on the basis of transversality condition by observing that the density of the linear momentum can be separated in a similar way [22, 33]. The separation of the SAM from the orbital angular momentum is gauge-independent. The SAM per photon is given by [4, 9, 32]  $\mathbf{S} = -i\hbar \int \mathbf{E}^* \times \mathbf{E} d\Omega$ . By virtue of the equality [4]  $\mathbf{a} \times \mathbf{b} = i\mathbf{a}^T \hat{\boldsymbol{\Sigma}} \mathbf{b}$ , where the superscript “T” denotes the transpose, it turns into

$$\mathbf{S} = \hbar \int \mathbf{E}^\dagger \hat{\boldsymbol{\Sigma}} \mathbf{E} d\Omega, \quad (1)$$

which has a quantum-mechanical interpretation as the expectation value of operator  $\hat{\mathbf{S}} = \hbar \hat{\boldsymbol{\Sigma}}$  in the state represented by the wave function  $\mathbf{E}(\mathbf{k})$ . Factorizing  $\mathbf{E}(\mathbf{k})$  into  $\mathbf{E}(\mathbf{k}) = \mathbf{e}(\mathbf{k})E(\mathbf{k})$ , where  $\mathbf{e}(\mathbf{k})$  is a complex unit vector determining the state of polarization and satisfying  $\mathbf{e}^\dagger \mathbf{e} =$

1,  $E(\mathbf{k})$  is generally a complex scalar function representing the amplitude and satisfying  $\int |E|^2 d\Omega = 1$ , and substituting into Eq. (1), one obtains  $\mathbf{S} = \hbar \int \mathbf{e}^\dagger \hat{\Sigma} \mathbf{e} |E|^2 d\Omega$ . This shows that the SAM per photon in a plane wave is [32]

$$\mathbf{s} = \hbar \mathbf{e}^\dagger \hat{\Sigma} \mathbf{e}. \quad (2)$$

### A. From transversality to quasi-unitary transformation

The transversality means that the three-component polarization vector  $\mathbf{e}$  is in reality a two-dimensional vector [31]. As a result,  $\mathbf{e}$  can be expanded in terms of a real linearly-polarized basis as

$$\mathbf{e} = \alpha_u \mathbf{u} + \alpha_v \mathbf{v} \equiv m_3 \tilde{\alpha}_3, \quad (3)$$

where the real unit vectors  $\mathbf{u}$  and  $\mathbf{v}$  are the base vectors that form a right-handed triad with the wave vector obeying [34]

$$\mathbf{u}^\dagger \mathbf{v} = 0, \quad (4a)$$

$$\mathbf{u} \times \mathbf{v} = \mathbf{w}, \quad (4b)$$

the unit vector  $\mathbf{w} = \frac{\mathbf{k}}{k}$  denotes the direction of the wave vector, and  $\tilde{\alpha}_3 = \begin{pmatrix} \alpha_u \\ \alpha_v \end{pmatrix}$  is a complex two-component unit spinor satisfying  $\tilde{\alpha}_3^\dagger \tilde{\alpha}_3 = 1$ . In Eq. (3) the base vectors  $\mathbf{u}$  and  $\mathbf{v}$  of the polarization basis constitute the 3-by-2 matrix [31]  $m_3 = \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix}$  that satisfies

$$m_3^\dagger m_3 = 1 \quad (5)$$

when Eq. (4) is considered. With the polarization basis  $m_3$ , the spinor  $\tilde{\alpha}_3$  plays the role of determining [4] the state of polarization and will be referred to as the polarization spinor. The meaning of subscript “3” will be clear shortly.

The polarization basis  $m_3$  is a QUT in the following sense. Firstly, it operates on a unit spinor and yields a unit vector as Eq. (3) shows. Secondly, one readily obtains  $\tilde{\alpha}_3 = m_3^\dagger \mathbf{e}$  from Eq. (3) by making use of Eq. (5). This shows that the 2-by-3 matrix  $m_3^\dagger$  operates on a unit vector and yields a unit spinor, indicating the following property:

$$m_3 m_3^\dagger = 1. \quad (6)$$

As a matter of fact, direct multiplication gives  $m_3 m_3^\dagger = 1 - \mathbf{w} \mathbf{w}^T$ . Because the matrix  $m_3^\dagger$  always operates on the polarization vector  $\mathbf{e}$  that is perpendicular to  $\mathbf{w}$ ,  $m_3 m_3^\dagger$  reduces to 1. That is to say, Eq. (6) has taken the transversality condition into account.  $m_3^\dagger$  is the Moore-Penrose pseudo-inverse [35] of  $m_3$ . Eqs. (5) and (6) express the quasi unitarity of the matrices  $m_3$  and  $m_3^\dagger$ . In this regard, we will term as the Jones space the space of all the unit polarization spinors on which  $m_3$  is defined. Correspondingly, we will term as the Pancharatnam space the space of all the unit polarization vectors on which  $m_3^\dagger$  is defined, upon considering that Pancharatnam [36] made the first investigation into the physical significance of the phase of polarization vector by exploring the interference between two non-orthogonal polarization vectors. The QUT's  $m$  and  $m^\dagger$  relate these two spaces to each other.

## B. Spin operator defined on the Jones space

Now we are ready to show that the spin operator on the Jones space is expressible in terms of the Pauli matrix. Substituting Eq. (3) into Eq. (2) and considering Eq. (4), one arrives at

$$\mathbf{s} = \hbar \tilde{\alpha}_3^\dagger \hat{\gamma}_3 \tilde{\alpha}_3, \quad (7)$$

where  $\hat{\gamma}_3 = m_3^\dagger \hat{\Sigma} m_3 = \mathbf{w} \hat{\sigma}_3$ ,  $\hat{\sigma}_3 = m_3^\dagger (\mathbf{w}^T \hat{\Sigma}) m_3$  is one of the three Pauli matrices [37], and  $\mathbf{w}^T \hat{\Sigma}$  means the inner product of the matrix vector  $\hat{\Sigma}$  and the unit wave vector  $\mathbf{w}$ . The spin operator  $\hat{\Sigma}$  that is defined on the Pancharatnam space is transformed by the QUT into  $\hat{\gamma}_3$  that is defined on the Jones space. This shows that  $\hat{\Sigma}$  is equivalent to  $\mathbf{w}(\mathbf{w}^T \hat{\Sigma})$  when the QUT is taken into account. The photon spin is thus always along  $\mathbf{w}$ . Remarkably, contrary to what might be expected [26, 27, 30], the spin operator  $\hat{\gamma}$  contains only one of the Pauli matrices,  $\hat{\sigma}_3$ . It describes exactly the fact that the spin has only two eigen states of eigen values  $\pm \hbar$  [38]. In addition, its Cartesian components commute with one another, in complete agreement with van Enk and Nienhuis' conclusion [10, 11].

It is the real-valuedness of the polarization basis  $m_3$  that makes the spin operator on the Jones space have the form of Pauli matrix  $\hat{\sigma}_3$ . This is why we adopt the subscript “3” to denote that polarization basis. As we know, the change of polarization basis is represented

by a unitary transformation [38]. Let us first consider the following unitary transformation,

$$U_2 = \exp\left(-i\frac{\pi}{4}\hat{\sigma}_2\right).$$

Inserting the identity  $U_2^\dagger U_2$  into Eq. (3) and letting

$$m_1 = m_3 U_2^\dagger = \left( \frac{\mathbf{u}+i\mathbf{v}}{\sqrt{2}} \quad \frac{\mathbf{v}+i\mathbf{u}}{\sqrt{2}} \right), \quad (8a)$$

$$\tilde{\alpha}_1 = U_2 \tilde{\alpha}_3, \quad (8b)$$

one has for the same polarization vector,

$$\mathbf{e} = m_1 \tilde{\alpha}_1. \quad (9)$$

The complex unit vectors  $(\mathbf{u} + i\mathbf{v})/\sqrt{2}$  and  $(\mathbf{v} + i\mathbf{u})/\sqrt{2}$  in Eq. (8a) describe the two orthogonal circular polarizations. They form the new polarization basis  $m_1$ , which is also a QUT. Substituting Eq. (9) into Eq. (2) yields

$$\mathbf{s} = \hbar \tilde{\alpha}_1^\dagger \hat{\gamma}_1 \tilde{\alpha}_1, \quad (10)$$

where  $\hat{\gamma}_1 = m_1^\dagger \hat{\Sigma} m_1 = \mathbf{w} \hat{\sigma}_1$ ,  $\hat{\sigma}_1 = m_1^\dagger (\mathbf{w}^T \hat{\Sigma}) m_1$ , showing that the spin operator  $\hat{\Sigma}$  on the Pancharatnam space is transformed by  $m_1$  into  $\hat{\gamma}_1$ . As is expected, the spin operator  $\hat{\gamma}_1$  on the Jones space has only component along  $\mathbf{w}$  and contains only one of the Pauli matrices,  $\hat{\sigma}_1$ .

Now we consider a second polarization-basis change represented by another unitary transformation  $U_3 = \exp(-i\frac{\pi}{4}\hat{\sigma}_3)$ . Inserting the identity  $U_3^\dagger U_3$  into Eq. (9) and letting

$$m_2 = m_1 U_3^\dagger = \left( \frac{\mathbf{v}-i\mathbf{u}}{\sqrt{2}} e^{i\frac{3\pi}{4}} \quad \frac{\mathbf{u}+i\mathbf{v}}{\sqrt{2}} e^{i\frac{\pi}{4}} \right), \quad (11a)$$

$$\tilde{\alpha}_2 = U_3 \tilde{\alpha}_1, \quad (11b)$$

one has another expression for the polarization vector,

$$\mathbf{e} = m_2 \tilde{\alpha}_2. \quad (12)$$

The matrix  $m_2$ , again a QUT, represents a third polarization basis that consists of a pair of complex rectilinear base vectors. Upon substituting Eq. (12) into Eq. (2), one gets

$$\mathbf{s} = \hbar \tilde{\alpha}_2^\dagger \hat{\gamma}_2 \tilde{\alpha}_2,$$

where  $\hat{\gamma}_2 = m_2^\dagger \hat{\Sigma} m_2 = \mathbf{w} \hat{\sigma}_2$  and  $\hat{\sigma}_2 = m_2^\dagger (\mathbf{w}^T \hat{\Sigma}) m_2$ . Again, the transformed spin operator  $\hat{\gamma}_2$  from  $\hat{\Sigma}$  by  $m_2$  is along  $\mathbf{w}$  and contains only one of the Pauli matrices,  $\hat{\sigma}_2$ .

In summary of this section we see that in a particular polarization basis  $m = \begin{pmatrix} \boldsymbol{\varepsilon}_1 & \boldsymbol{\varepsilon}_2 \end{pmatrix}$ , the polarization vector is expressed as

$$\mathbf{e} = m \tilde{\alpha}, \quad (13)$$

where  $\tilde{\alpha}$  is the polarization spinor associated with  $m$ , the base vectors  $\boldsymbol{\varepsilon}_1$  and  $\boldsymbol{\varepsilon}_2$  obey

$$\boldsymbol{\varepsilon}_1^\dagger \boldsymbol{\varepsilon}_2 = 0, \quad (14a)$$

$$\boldsymbol{\varepsilon}_1 \times \boldsymbol{\varepsilon}_2 = \mathbf{w}, \quad (14b)$$

which guarantees that  $m$  and  $m^\dagger$  satisfy

$$m^\dagger m = m m^\dagger = 1 \quad (15)$$

and act as the QUT's connecting the Jones and Pancharatnam spaces. The spin operator  $\hat{\Sigma}$  on the Pancharatnam space is transformed into  $\hat{\gamma} = m^\dagger \hat{\Sigma} m = \mathbf{w} \hat{\gamma}$  on the Jones space, where

$$\hat{\gamma} = m^\dagger (\mathbf{w}^T \hat{\Sigma}) m \quad (16)$$

is an Hermitian unitary matrix satisfying  $\hat{\gamma}^2 = 1$ . This shows that the spin operator on the Jones space is expressed by a single Hermitian unitary matrix and is always along the direction of the wave vector. Denoting the eigen spinors of operator  $\hat{\gamma}$  by  $\tilde{\alpha}_\pm$  satisfying  $\hat{\gamma} \tilde{\alpha}_\pm = \pm \tilde{\alpha}_\pm$  and noticing Eq. (15), one then has  $(\mathbf{w}^T \hat{\Sigma}) \mathbf{e}_\pm = \pm \mathbf{e}_\pm$ , where  $\mathbf{e}_\pm = m \tilde{\alpha}_\pm$ . This means that the eigen spinors of  $\hat{\gamma}$  correspond to the eigen vectors of operator  $\mathbf{w}^T \hat{\Sigma}$  via Eq. (13). If the polarization basis is changed according to  $m' = m U^\dagger$  and  $\tilde{\alpha}' = U \tilde{\alpha}$  with a unitary transformation  $U$ , the operator  $\hat{\gamma}$  is changed as

$$\hat{\gamma}' = U \hat{\gamma} U^\dagger. \quad (17)$$

Consequently, the Hermitian unitary matrix in the spin operator  $\hat{\gamma}$  takes different forms in different polarization bases.

### III. ONE-TO-ONE CORRESPONDENCE BETWEEN THE SO(3) AND SU(2) ROTATIONS

Berry [30] once observed that the SAM is invariant when the polarization vector is rotated about the wave vector. This is the case because it is just in the direction of the wave vector.

In this section we will show that a  $\text{SO}(3)$  rotation of the polarization vector about the wave vector corresponds, via the QUT, to a  $\text{SU}(2)$  rotation of the polarization spinor about the same axis through the same angle, a relation that is quite different from what is known in the literature [39].

Consider a polarization vector  $\mathbf{e}$  that is given by Eq. (13). A  $\text{SO}(3)$  rotation  $R(\Phi\mathbf{w}) = \exp\{-i(\mathbf{w}^T\hat{\Sigma})\Phi\}$  about  $\mathbf{w}$  through an angle  $\Phi$  transforms  $\mathbf{e}$  into

$$\mathbf{e}' = R(\Phi\mathbf{w})\mathbf{e} = R(\Phi\mathbf{w})m\tilde{\alpha}. \quad (18)$$

This can be interpreted as rotating the polarization basis about  $\mathbf{w}$ ,  $m' = R(\Phi\mathbf{w})m$ , with the polarization spinor remaining unchanged. Since  $\mathbf{w}^T\hat{\Sigma}$  commutes with  $R(\Phi\mathbf{w})$ , it follows from Eq. (16) that the spin operator on the Jones space is invariant under a rotation of the polarization basis about  $\mathbf{w}$ ,

$$\hat{\gamma}' = m'^{\dagger}(\mathbf{w}^T\hat{\Sigma})m' = m^{\dagger}(\mathbf{w}^T\hat{\Sigma})m = \hat{\gamma}.$$

As a matter of fact, the two conditions in Eq. (14) do not uniquely determine the polarization basis up to such a rotation [38]. The  $\text{SO}(3)$  rotation operator can be written as [39]

$$R(\Phi\mathbf{w}) = \cos \Phi - i(\mathbf{w}^T\hat{\Sigma}) \sin \Phi + (1 - \cos \Phi)\mathbf{w}\mathbf{w}^T.$$

Substituting it into Eq. (18) and noticing that the base vectors in  $m$  are perpendicular to  $\mathbf{w}$ , one has

$$\mathbf{e}' = \{\cos \Phi - i(\mathbf{w}^T\hat{\Sigma}) \sin \Phi\}m\tilde{\alpha}.$$

By making use of Eqs. (15) and (16), one may rewrite it as

$$\mathbf{e}' = m \exp(-i\hat{\gamma}\Phi)\tilde{\alpha}.$$

This can be reinterpreted as rotating the polarization spinor about  $\mathbf{w}$ ,

$$\tilde{\alpha}' = \exp(-i\hat{\gamma}\Phi)\tilde{\alpha}, \quad (19)$$

the generator being the spin operator, with the polarization basis remaining unchanged. Because the generator of this  $\text{SU}(2)$  rotation is without the factor  $\frac{1}{2}$  that we encounter in the case of electrons, the rotation angle is the same as the  $\text{SO}(3)$  rotation. This completes our proof.



The invariance of the SAM under the rotation of the polarization vector or the polarization spinor about  $\mathbf{w}$  means that the spin is different from the polarization. Since the polarization state of a completely polarized plane wave can be exactly described by the Stokes vector, we will explore in the next the relation of the SAM with the Stokes vector.

#### IV. SAM IS THE COMPONENT OF THE STOKES VECTOR ALONG $\mathbf{w}$

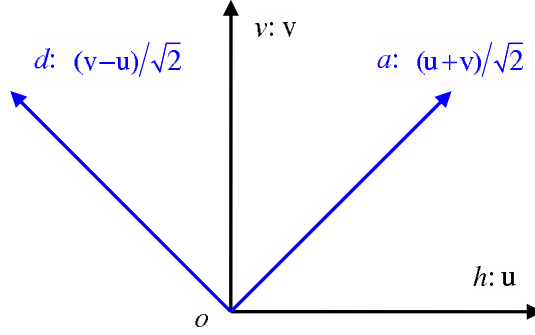


FIG. 1: Schematic diagram illustrating the connection of the base polarization vectors that form the QUT  $m_3$  and  $m_2$  to the orientations of the linear polarizer for measuring the Stokes parameters  $p_1$  and  $p_2$ .

The three components of the Stokes vector  $\mathbf{p}$  are defined by [20]

$$p_1 = \frac{I_h - I_v}{I_h + I_v}, \quad p_2 = \frac{I_a - I_d}{I_a + I_d}, \quad p_3 = \frac{I_r - I_l}{I_r + I_l}, \quad (20)$$

where  $I_h$ ,  $I_v$ ,  $I_a$ , and  $I_d$  are the intensities of the wave measured through the corresponding orientations of the linear polarizer as is shown in Fig. 1,  $I_r$  and  $I_l$  are respectively the intensities of the right-handed and left-handed circularly polarized components in the wave. The subscripts “h” and “v” mean the horizontal and vertical orientations of the polarizer, respectively, that correspond to the base vectors  $\mathbf{u}$  and  $\mathbf{v}$  of the polarization basis  $m_3$ . Similarly, the subscripts “d” and “a” mean that the respective orientations of the polarizer correspond to the base vectors  $\frac{\mathbf{v}-\mathbf{u}}{\sqrt{2}}e^{i\frac{3\pi}{4}}$  and  $\frac{\mathbf{u}+\mathbf{v}}{\sqrt{2}}e^{i\frac{\pi}{4}}$  of the polarization basis  $m_2$ . With the help of polarization basis  $m_3$ ,  $p_1$  is written in terms of the polarization spinor  $\tilde{\alpha}_3$  as

$$p_1 = \tilde{\alpha}_3^\dagger \hat{\sigma}_1 \tilde{\alpha}_3. \quad (21)$$

In addition, the structure of the polarization basis  $m_2$  allows us to express  $p_2$  as  $p_2 =$

$-\tilde{\alpha}_2^\dagger \hat{\sigma}_1 \tilde{\alpha}_2$ . Noticing  $\tilde{\alpha}_2 = U_3 \tilde{\alpha}_1$  and  $\tilde{\alpha}_1 = U_2 \tilde{\alpha}_3$ , it turns out to be

$$p_2 = \tilde{\alpha}_3^\dagger \hat{\sigma}_2 \tilde{\alpha}_3. \quad (22)$$

Furthermore,  $p_3$  can be expressed as  $p_3 = \tilde{\alpha}_1^\dagger \hat{\sigma}_1 \tilde{\alpha}_1$  in accordance with the polarization basis  $m_1$ . This is exactly the SAM (in units of  $\hbar$ ) as Eq. (10) shows. A comparison between Eqs. (10) and (7) leads to

$$p_3 = \tilde{\alpha}_3^\dagger \hat{\sigma}_3 \tilde{\alpha}_3. \quad (23)$$

Collecting Eqs. (21)-(23) all together, we find that the Stokes vector can be expressed in the polarization basis  $m_3$  as

$$\mathbf{p} = \tilde{\alpha}_3^\dagger \hat{\boldsymbol{\sigma}} \tilde{\alpha}_3, \quad (24)$$

in terms of the Pauli vector  $\hat{\boldsymbol{\sigma}}$  [40, 41].

In view of Eq. (24), the Pauli vector should be regarded as the polarization operator. This is why the polarization of photons can be compared [25] to the spin of electrons. Since the Stokes vector is invariant under the change of polarization basis, the detailed form of the Pauli vector is dependent on the choice of polarization basis. If the polarization basis is changed according to  $\tilde{\alpha}' = U \tilde{\alpha}_3$  with  $U$  a unitary transformation, the Stokes vector appears to be  $\mathbf{p} = \tilde{\alpha}'^\dagger U \hat{\boldsymbol{\sigma}} U^\dagger \tilde{\alpha}'$ . Letting be  $\hat{\boldsymbol{\sigma}}'$  the Pauli vector in the new polarization basis,

$$\hat{\boldsymbol{\sigma}}' = U \hat{\boldsymbol{\sigma}} U^\dagger, \quad (25)$$

one has  $\mathbf{p} = \tilde{\alpha}'^\dagger \hat{\boldsymbol{\sigma}}' \tilde{\alpha}'$ . In a word, denoting respectively by  $\hat{\boldsymbol{\sigma}}$  and  $\tilde{\alpha}$  the Pauli vector and the polarization spinor in a particular polarization basis, the Stokes vector is given by

$$\mathbf{p} = \tilde{\alpha}^\dagger \hat{\boldsymbol{\sigma}} \tilde{\alpha}. \quad (26)$$

We have shown that the third component of the Stokes vector is equal to the SAM. Now that the SAM is in the direction of the wave vector, it is exactly the component of the Stokes vector along the wave vector. This is easily proven. Suppose that the spin operator on the Jones space is given by

$$\hat{\gamma} = \mathbf{w}^T \hat{\boldsymbol{\sigma}} \quad (27)$$

with  $\hat{\boldsymbol{\sigma}}$  the Pauli vector in a particular polarization basis. Eq. (27) guarantees that the component of the Stokes vector along  $\mathbf{w}$ ,  $p_3 = \tilde{\alpha}^\dagger (\mathbf{w}^T \hat{\boldsymbol{\sigma}}) \tilde{\alpha}$ , is invariant under the rotation of the polarization spinor about  $\mathbf{w}$  by virtue of Eq. (19), the same as the SAM is. So the

component of the Stokes vector along  $\mathbf{w}$  is the SAM. Jauch and Rohrlich [7] once found that the SAM is equal to one component of the Stokes vector in a second quantization theory of the radiation field. Unfortunately, their result received little attention.

## V. CONCLUDING REMARKS

In conclusion, we have shown that the photon spin operator on the Jones space is given by  $\hat{\gamma} = \mathbf{w}\hat{\gamma}$  in terms of an Hermitian unitary matrix  $\hat{\gamma}$ . It is in the direction of the wave vector. Its Cartesian components are commutative with one another. This operator is obtained through transforming the spin operator  $\hat{\Sigma}$  that is defined on the Pancharatnam space and satisfies the appropriate commutation relations by making use of a QUT that is associated with a particular polarization basis. The form of matrix  $\hat{\gamma}$  depends on the choice of the polarization basis. In the commonly used polarization bases denoted by  $m_1$ ,  $m_2$ , and  $m_3$ ,  $\hat{\gamma}$  takes the form of the Pauli matrices  $\hat{\sigma}_1$ ,  $\hat{\sigma}_2$ , and  $\hat{\sigma}_3$ , respectively. On the basis of this theory, we found that a SO(3) rotation of the polarization vector about the wave vector corresponds to a SU(2) rotation of the polarization spinor through the same angle about the same axis. It is very interesting to note that the generator of the former rotation is the component of spin operator  $\hat{\Sigma}$  along the wave vector and that of the latter rotation is the spin operator  $\hat{\gamma}$  itself. Furthermore, the SAM is just the component of the Stokes vector along the wave vector.

The theory advanced here for the photon spin on the Jones space is a probabilistic one that is compatible with the quantum mechanical description [4] of a photon state. Upon transforming from the Pancharatnam space to the Jones space by the QUT, one arrives at the two-component polarization spinor, which is much suitable to deal with the angular momentum problem of the radiation field. It is noted that the two conditions in Eq. (4) or Eq. (14) leave the polarization basis undetermined up to a rotation about the wave vector. It is the degree of freedom of this rotation in defining the QUT [31] that makes it necessary for us to introduce a fixed unit vector to represent a vector electromagnetic beams. That vector plays a very important role in describing the orbital angular momentum of light beams [32] and in understanding the spin Hall effect of light [42]. A general formalism for the spin and orbital angular momentum on the Jones space is under preparation.

## Acknowledgments

CFL would like to thank Masud Mansuripur of the University of Arizona for his helpful discussions. This work was supported in part by the National Natural Science Foundation of China (60877055 and 60806041), the Shanghai Educational Development Foundation (2007CG52), and the Shanghai Leading Academic Discipline Project (S30105).

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$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

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